HFOFO study
Transverse Beam Dynamics

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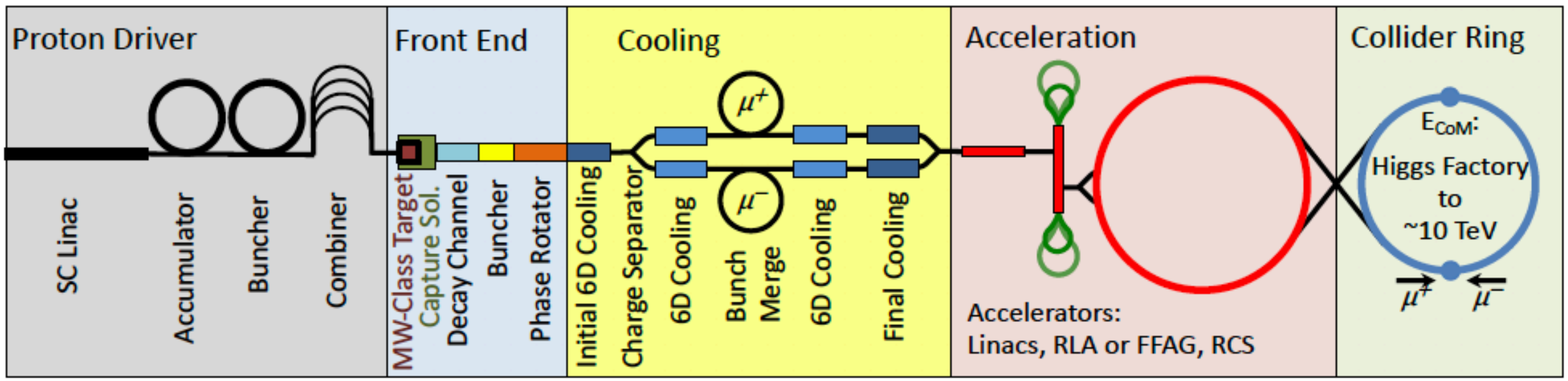
Discussion item

- Overview of Muon Collider design
- Concept of Ionization cooling
- Basic of transverse beam dynamics

Overview of Muon Collider

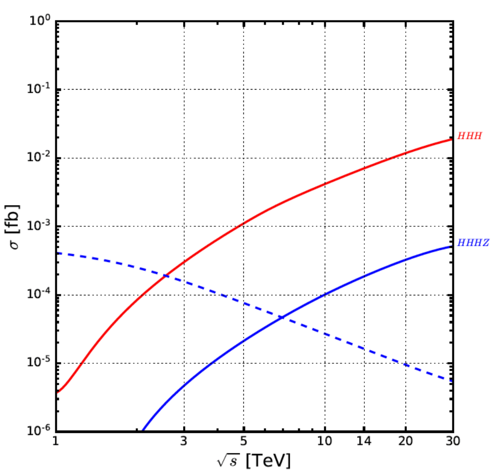
Proton beam based muon collider



- Goal collider luminosity

$$\mathscr{L} = \frac{N_{\mu^+} \bullet N_{\mu^-} \bullet f \bullet n_b}{4\pi \bullet \sigma_x \bullet \sigma_y} > 10^{34} \quad \text{cm}^{-2}\,\text{s}^{-1}$$

N: Num of μ per bunch ($\sim 10^{12}$)
f: Bunch revolution (~ 1000)
n_b: Repetition rate (~ 5 Hz)
σ: Beam spot size at collision ($\sim 10^{-4}$ cm)



Ex) $\mu\mu \rightarrow WW \rightarrow HHH$, VBF at 14 TeV

$\sigma = 7.1 \bullet 10^{-3}$ femto barns = $7.1 \bullet 10^{-18}$ barns = $7.1 \bullet 10^{-42}$ cm^{-2}

6 months full time beam operation

→ $t_{operation} \sim 1.56 \bullet 10^7$ s

nsvers $f_{\mu\mu\rightarrow HHH} \sim 7.1 \bullet 10^{-42} \times 1.56 \bullet 10^7 \times 10^{34} \sim 0.1$ event

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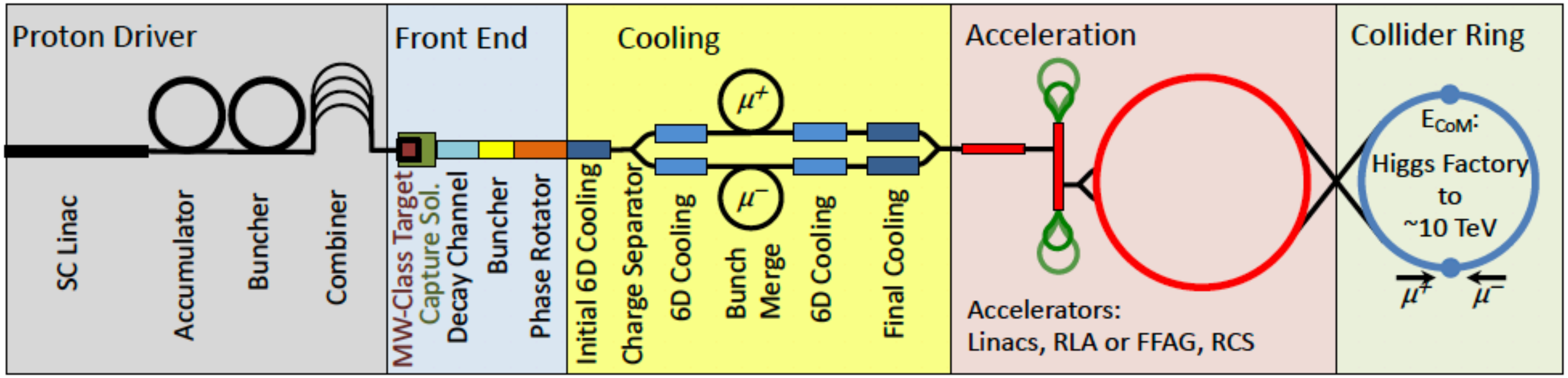


- Required muons after cooling, $N\sim10^{12}$, $\varepsilon_{t,n}\sim20$ mrad
  - Required proton beam: $\sim10^{15}$ protons/spill
  - Required $\mu/proton$: $\sim0.1\ \mu/p$
  - Acceptable μ loss in cooling (and acceleration): 0.01
- Other potential approaches of muon source without cooling
  - Muon pair production via $e^+e^-\rightarrow\mu^+\mu^-$ (LEMMA)
  - Ultracold muons through ionizing muonium and muonic atoms
    - Beam spot size extremely small, e.g. $\sigma\sim10^{-5}-10^{-6}$ cm
    - $\mu^\pm$ yield will be small, e.g. $N\sim10^{11}-10^{10}$

Overview of Muon Collider

- Intense proton beam strikes pion production target

$$p + A \rightarrow \pi^{\pm} + A'$$

- Pions are eventually decayed

$$\pi^{+} \rightarrow \mu^{+} + \nu_{\mu}$$
$$\pi^{-} \rightarrow \mu^{-} + \bar{\nu}_{\mu}$$

- Pion lifetime

$\tau = 26\gamma$ ns $\rightarrow \tau \bullet c = 7.8$ m ($\gamma = 1$)

- Muons are eventually decayed as well

$$\mu^{+} \rightarrow e^{+} + \nu_{e} + \bar{\nu}_{\mu}$$
$$\mu^{-} \rightarrow e^{-} + \bar{\nu}_{e} + \nu_{\mu}$$

- Muon lifetime

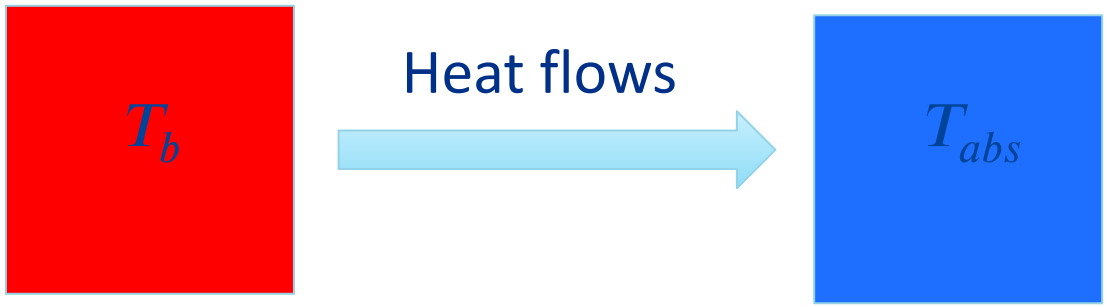
$\tau = 2.2\gamma$ μs $\rightarrow \tau \bullet c = 659.5$ m ($\gamma = 1$)

Challenge in producing low emittance muon beam

- Initial muon phase space is too large to achieve the goal luminosity
  - Phase space cooling is required
    - Initial muon phase space is similar as basketball
    - Required muon phase space after cooling is sub-millimeter
- Muons have a finite lifetime
  - Need fast cooling scheme
    - Ionization cooling

Beam cooling

- Heat flows from warm object to cold object



Heat flows when

$T_b > T_{abs}$

Equilibrium condition

$T_b \sim T_{abs}$

$$\frac{dQ}{dt} = \dot{Q} = h \bullet A \bullet \left(T_b - T_{abs}\right)$$

Heat transfer rate

$$\rightarrow \frac{dQ}{ds} = Q' = h \bullet A \bullet \left(T_b - T_{abs}\right)$$

s is a path length

<u>Since $h \bullet A$ is constant, if ΔT is large, $\dot{Q}$ should be proportionally large</u>

↓

If ΔT is near zero (temperature reaches equilibrium), $\dot{Q}$ is small

Beam cooling

- Beam temperature flows into absorber temperature

Beam

$$\frac{\Delta p_{\perp}^2}{2m} = \frac{kT_b}{2}$$

q (momentum)-transfer via coulomb interaction

Absorber

$$\frac{kT_{abs}}{2}$$

Heat flows when

$$T_b > T_{abs}$$

Equilibrium condition

$$T_b \sim T_{abs}$$

How to effectively generate high transverse momentum in a beam?

- Transverse momentum is maximized at beam waist
- Stronger focusing generates higher transverse momentum (though it is not always true if beam has a large dispersion)

Beam cooling

- Beam temperature flows into absorber temperature

Beam

$$\frac{\Delta p_\perp^2}{2m} = \frac{kT_b}{2}$$

q (momentum)-transfer via coulomb interaction

Absorber

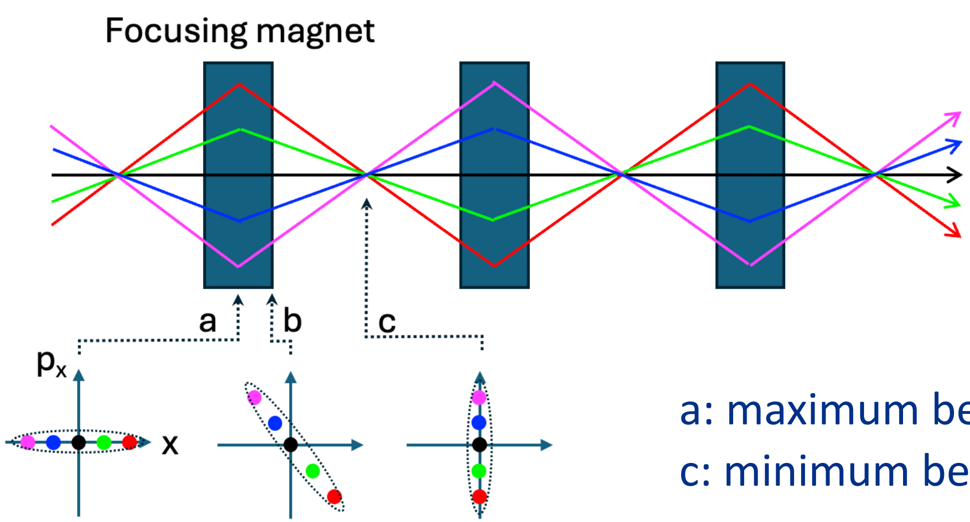
$$\frac{kT_{abs}}{2}$$

Heat flows when

$$T_b > T_{abs}$$

Equilibrium condition

$$T_b \sim T_{abs}$$



Put absorber at point "c"

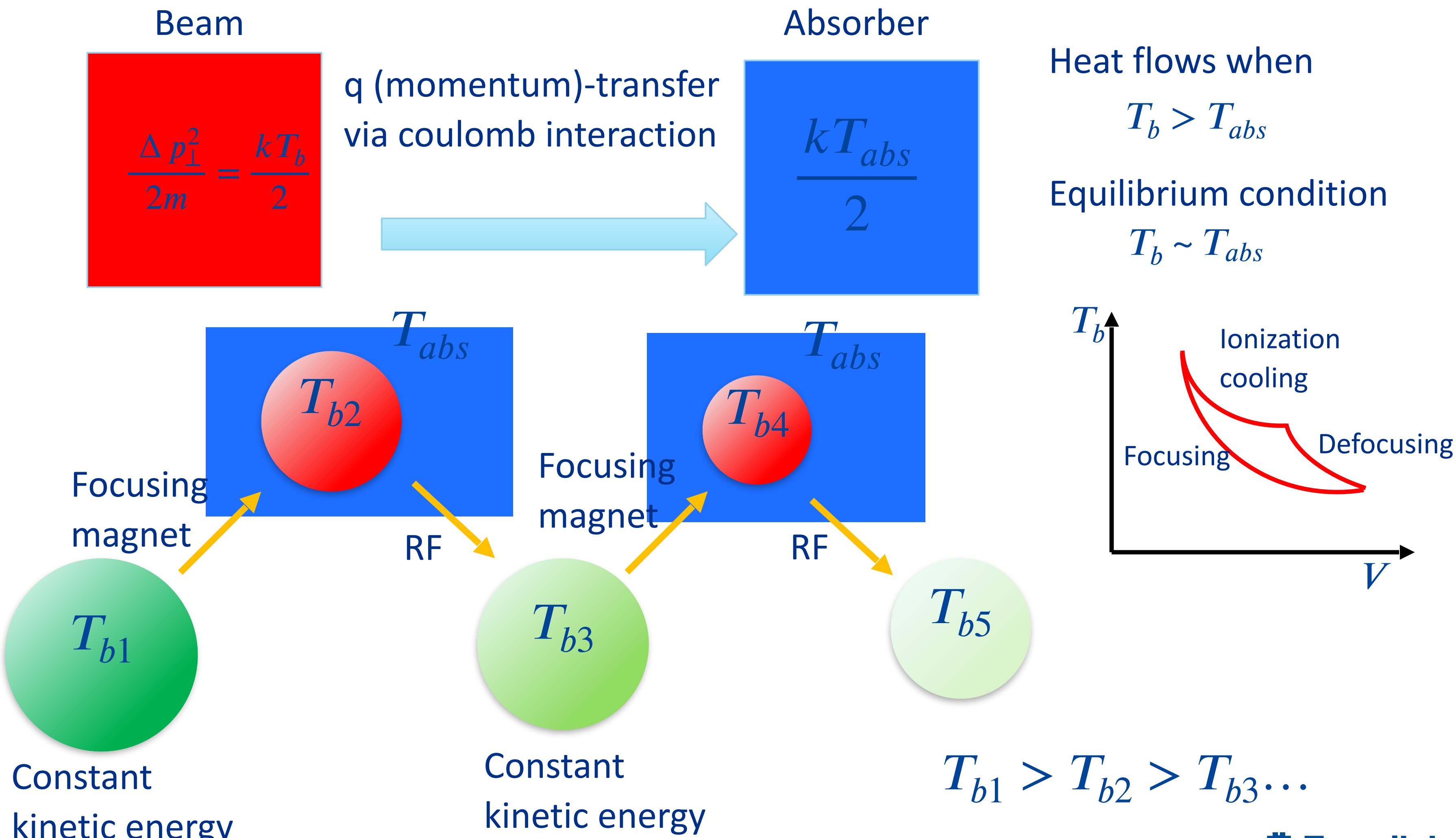
a: maximum beta function

c: minimum beta function

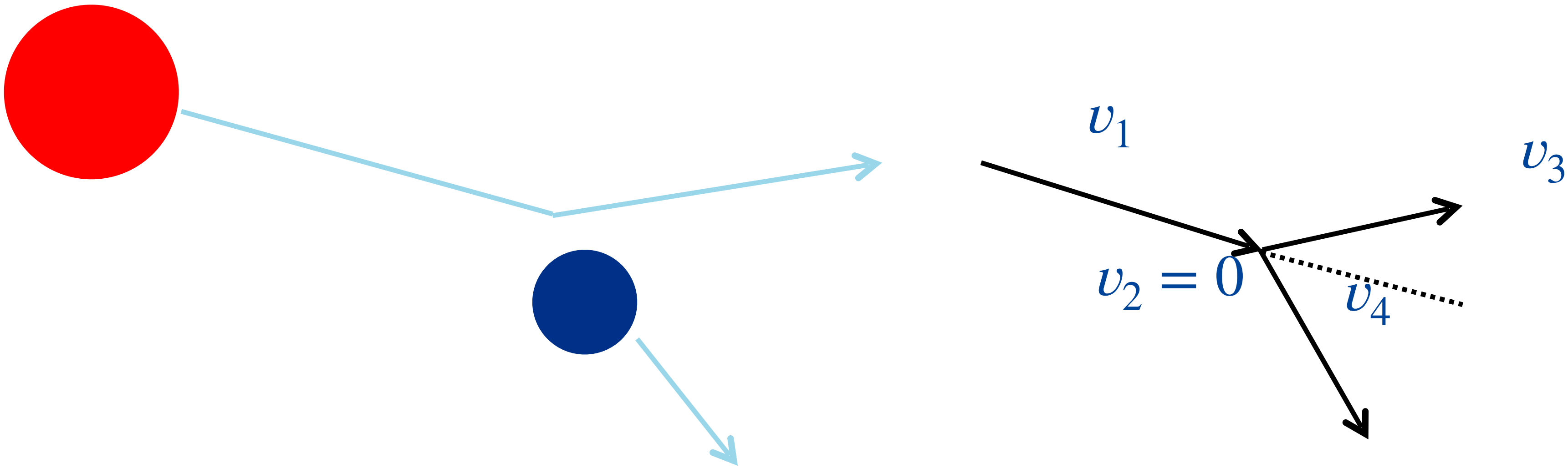
Transverse phase space is rotated along beam path

Beam cooling

- Beam temperature flows into absorber temperature



Energy transfer process (intuitive approach)



From energy conservation:

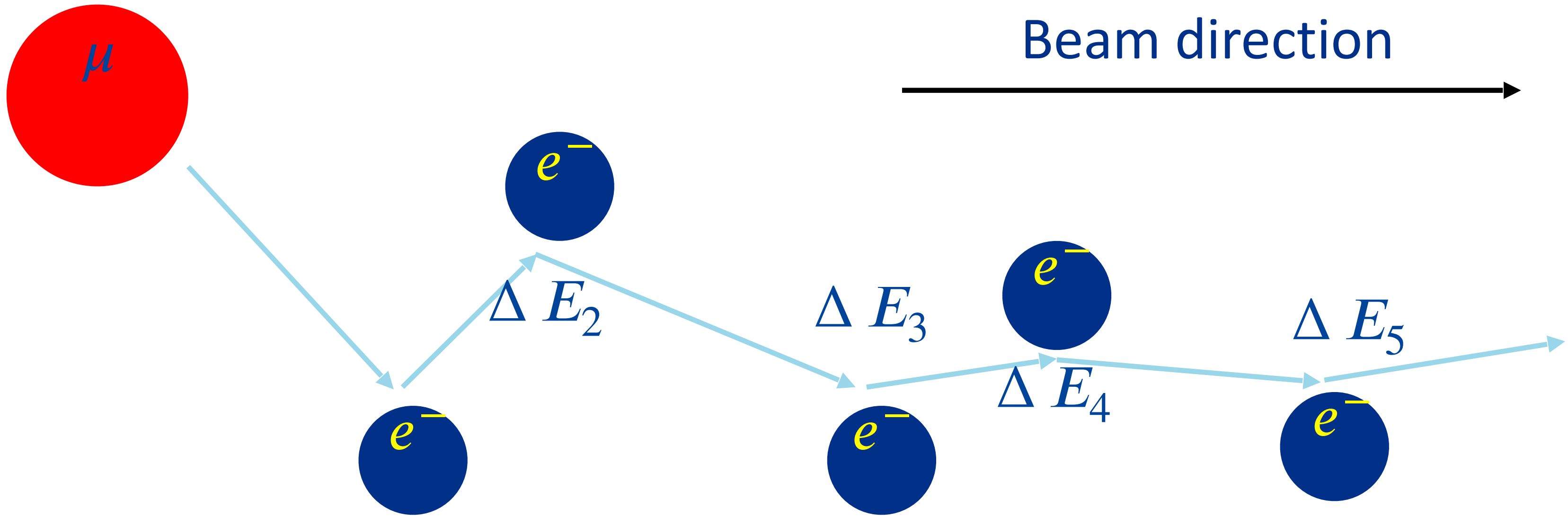
$$\frac{1}{2}m_b v_1^2 = \frac{1}{2}m_b v_3^2 + \frac{1}{2}m_t v_4^2 \rightarrow E_1 - E_3 = \Delta E = E_4$$

This is a transferred energy

The phase space is cooled by: $\Delta E / E$

(Inelastic collision: $E_1 - E_3 = \Delta E = E_4 + Q$)

Energy transfer in ionization cooling (intuitive approach)



$$\Delta E_1 = I + \delta E$$

Total energy loss per unit length:

$$\frac{dE}{dx} \sim K \frac{Z}{A} \frac{1}{\beta^2} \left[\frac{1}{2} ln \frac{2 m_e c^2 \beta^2 \gamma^2 W_{max}}{I^2} - \beta^2 \right] \sim n_c \bullet \bar{\Delta} E \sim n_c \bullet W$$

Bethe equation

Pure cooling rate

- As an example, in liquid H2 (LH2), $\frac{dE}{dx}$~320 keV/cm at p_μ~200 MeV/c
- It is known from measurements, W_{LH2}~20 eV, thus
  $n_c = \frac{320 \bullet 10^3}{20} = 16{,}000$ /cm

$$\sigma_{H \to H^+} \sim n_c \bullet \left(N_a \bullet \frac{\rho}{A}\right)^{-1} \sim 10^{-19}\ \text{cm}^{-2}$$

- Cooling rate is given,
  - $\lambda_{6D}^{-1} = \frac{\left(\frac{dE}{dx}\right)}{\beta^2 \bullet E}$ (we will derive this later)
  - In the case of LH2,
    - $\lambda_{6D}^{-1} \sim 320 \bullet 10^3 / 177 \bullet 10^6 \sim 1.8 \bullet 10^{-3}$ /cm
    - $\frac{\varepsilon_{6D,\,initial}}{\varepsilon_{6D,final}} = exp\left(-L \bullet \lambda_{6D}^{-1}\right)$, if $L = 100$ meter, $\frac{\varepsilon_{6D,\,initial}}{\varepsilon_{6D,final}} = 1.5 \bullet 10^{-8}$ (our goal is 10^{-6}!)
  - In reality, there is a heating term due to multiple scattering which will be discussed later

Transverse beam dynamics

- Beam dynamics is analogous to the dynamics of a simple harmonic motion
  - Solving harmonic oscillator via canonical transformation

$$\begin{cases} H = E = \frac{p^2}{2m} + \frac{1}{2}kq^2 \\ \dot{p} = -\frac{dH}{dq} = -kq \end{cases} \rightarrow \begin{cases} q = \sqrt{\frac{2E}{m\omega^2}}\cos\left(\omega t + \varphi_0\right) \\ p = \sqrt{2mE}\sin\left(\omega t + \varphi_0\right) \end{cases}$$

Or $\mathscr{H}(P,Q) = E = \omega \bullet P,\ \dot{Q} = \omega = \sqrt{\frac{k}{m}}$

$$\begin{cases} q = \sqrt{\frac{2E}{m\omega^2}}\cos\left(\omega t + \phi_0\right) \\ p = \sqrt{2mE}\sin\left(\omega t + \phi_0\right) \end{cases} \rightarrow \begin{cases} q = \sqrt{\frac{2P}{m\dot{Q}}}\cos\left(\omega t + \phi_0\right) \\ p = \sqrt{2m\dot{Q}P}\sin\left(\omega t + \phi_0\right) \end{cases}$$

Transverse beam dynamics

- Harmonic motion → Hill's equation (translate the differentiation from t to s)

$$\dot{p} = -\frac{dH}{dq} = -kq \rightarrow x'' + Kx = 0$$

Non-linear & coupling beam dynamics can be investigated through non-linear harmonic oscillators, which most likely uses a perturbation theory

Let us use following general solution for x

$$x = \sqrt{2\hat{\beta}_x(s)J_x}\cos\left[\hat{\mu}_x(s) + \phi_x\right]$$

Comparing $q = \sqrt{\frac{2P}{m\dot{Q}}}\cos(\omega t + \phi_0)$

$$P \rightarrow \frac{J_x}{m}, \dot{Q} \rightarrow \frac{1}{\hat{\beta}_x} = \hat{\mu}_x' \leftarrow (\hat{\mu}_x \rightarrow \omega t),\ \phi_x \rightarrow \phi_0,$$

Transverse beam dynamics

- Let us confirm
  - Differentiate x and x''

$$x' = \frac{dx}{ds} = \sqrt{\frac{J_x}{2\hat{\beta}_x(s)}}\hat{\beta}_x{}'(s)\cos\left[\hat{\mu}_x(s)+\phi_x\right] + \sqrt{2\hat{\beta}_x(s)J_x}\sin\left[\hat{\mu}_x(s)+\phi_x\right]\bullet\hat{\mu}_x{}'(s),$$

$$x'' = \frac{d^2x}{ds^2} = \sqrt{2\hat{\beta}_x(s)J_x}\left\{\hat{\mu}_x{}''(s) + \frac{\hat{\beta}_x{}'(s)}{\hat{\beta}_x(s)}\hat{\mu}_x{}'(s)\right\}\sin\left[\hat{\mu}_x(s)+\phi_x\right]$$

$$+\sqrt{2\hat{\beta}_x(s)J_x}\left\{-\frac{1}{4}\frac{\left(\hat{\beta}_x{}'(s)\right)^2}{\hat{\beta}_x^2} + \frac{1}{2}\frac{\hat{\beta}_x{}''(s)}{\hat{\beta}_x(s)} - \left(\hat{\mu}_x{}'\right)^2\right\}\cos\left[\hat{\mu}_x(s)+\phi_x\right].$$

Transverse beam dynamics

- Substitute into Hill's equation
  - Since right-handed-side of Hill's equation is zero, the coefficient of trigonal functions should be zero

$$\mu_x'' + \frac{\hat{\beta}_x'}{\hat{\beta}_x}\hat{\mu}_x' = 0 \rightarrow \left(\hat{\beta}_x \bullet \hat{\mu}_x'\right)' = 0 \rightarrow \boxed{\hat{\mu}_x' = \frac{1}{\hat{\beta}_x(s)},}$$

This is what we look into!

$$\cdots + \frac{1}{2}\frac{\hat{\beta}_x''}{\hat{\beta}_x} - \left(\hat{\mu}_x'\right)^2 + K_x = 0 \rightarrow 2\hat{\beta}_x\hat{\beta}_x'' - \left(\hat{\beta}_x'\right)^2 + 4K_x\hat{\beta}_x^2 = 4.$$

The second equation is often called envelop equation

Transverse beam dynamics

- Proposing further transformations

$$\hat{\alpha}_x = -\frac{1}{2}\hat{\beta}_x{}', \ \hat{\gamma}_x = \frac{1+\hat{\alpha}_x{}^2}{\hat{\beta}_x} \rightarrow \hat{\beta}_x\hat{\gamma}_x - \hat{\alpha}_x{}^2 = 1,$$

- Surprisingly (or expectedly), x and x' becomes much simple

$$x = \sqrt{2\hat{\beta}_x J_x}\cos\psi_x,$$

$$x' = \frac{\hat{\beta}_x{}'}{2\beta_x}\bullet\sqrt{2\hat{\beta}_x J_x}\cos\psi_x - \frac{\sqrt{2\hat{\beta}_x J_x}}{\hat{\beta}_x}\sin\psi_x = -\sqrt{\frac{2J_x}{\hat{\beta}_x}}\left(\sin\psi_x + \hat{\alpha}_x\cos\psi_x\right)$$

$$\left(= -\frac{x}{\hat{\beta}_x}\left[\tan\psi_x - \frac{\hat{\beta}_x{}'}{2}\right]\right).$$

- Envelop equation is also simplified

$$2\hat{\beta}_x\hat{\beta}_x{}'' - \left(\hat{\beta}_x{}'\right)^2 + 4K_x\hat{\beta}_x^2 = 4 \rightarrow \hat{\alpha}_x{}' = K_x\hat{\beta}_x - \frac{1}{\beta_x}\left[1+\hat{\alpha}_x{}^2\right] = K_x\hat{\beta}_x - \hat{\gamma}_x$$

Transverse beam dynamics

- We introduce a (new) canonical momentum

$$p_x \equiv \hat{\beta}_x x' + \hat{\alpha}_x x = -\sqrt{2\hat{\beta}_x J_x}\sin\psi_x .$$

$$x = \sqrt{2\hat{\beta}_x J_x}\cos\psi_x,$$

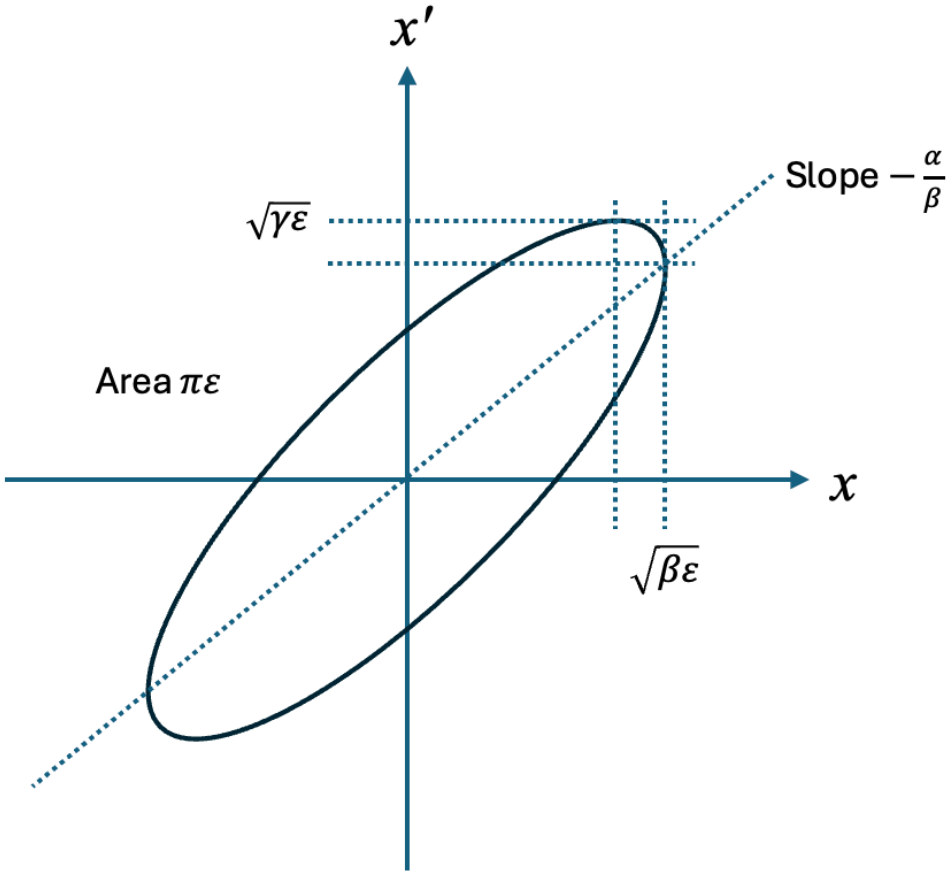
- Then,

$$\frac{1}{\hat{\beta}_x}\left\{\left(1+\hat{\alpha}_x^{\,2}\right)x^2 + 2\hat{\alpha}_x\hat{\beta}_x x x' + \hat{\beta}_x^{\,2}(x')^2\right\} \equiv \hat{\gamma}_x x^2 + 2\hat{\alpha}_x x x' + \hat{\beta}_x (x')^2 = 2.$$

$$\hat{\gamma}_x \equiv \frac{1}{\hat{\beta}_x}\left[1+\hat{\alpha}_x^{\,2}\right]$$

ε_x is called the Courant-Snyder invariant, or emittance

Courant-Snyder invariant (phase ellipse)



x and p_x form a circle with radius $\sqrt{\hat{\beta}_x \bullet \varepsilon_x}$

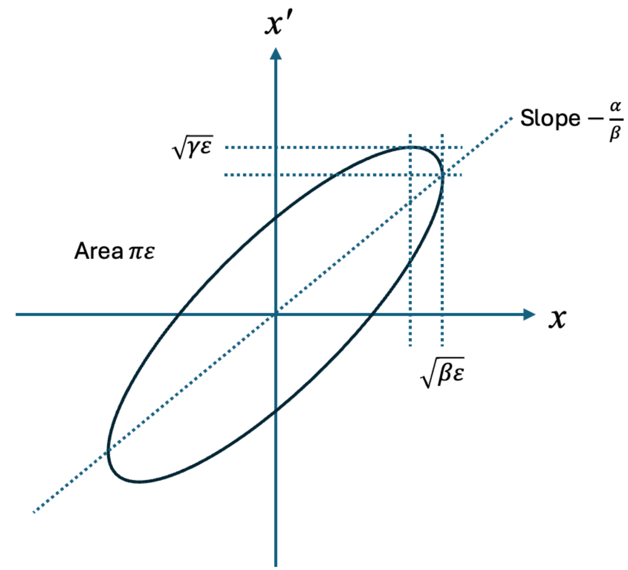
Summary

- We have a solution for particle motion using analogous to the simple harmonic motion

$$x = \sqrt{2\hat{\beta}_x J_x}\cos\psi_x,$$

$$x' = -\sqrt{\frac{2J_x}{\hat{\beta}_x}}\left(\sin\psi_x + \hat{\alpha}_x\cos\psi_x\right)$$

$$\frac{1}{\hat{\beta}_x} = \psi_x'$$



- Smaller β_x makes amplitude of x' larger while amplitude of x smaller
- How do we find β?
  - Can we solve β from envelop equation?

$$2\hat{\beta}_x\hat{\beta}_x'' - \left(\hat{\beta}_x'\right)^2 + 4K_x\hat{\beta}_x^2 = 4 \rightarrow \hat{\alpha}_x' = K_x\hat{\beta}_x - \frac{1}{\hat{\beta}_x}\left[1 + \hat{\alpha}_x^2\right] = K_x\hat{\beta}_x - \hat{\gamma}_x$$

Find $\hat{\beta}$,$\hat{\alpha}$,$\hat{\gamma}$(Twiss parameter) from phase space evolution

- Assume the density function is given, $\rho(x, x')$

$$\langle x \rangle = \int x \bullet \rho(x, x')dxdx', \ \langle x' \rangle = \int x' \bullet \rho(x, x')dxdx',$$

$$\sigma_x^2 = \int \left(x - \langle x \rangle\right)^2 \bullet \rho(x, x')dxdx',$$

$$\sigma_{x'}^2 = \int \left(x' - \langle x' \rangle\right)^2 \bullet \rho(x, x')dxdx',$$

$$\sigma_{xx'} = \int \left(x - \langle x \rangle\right)\left(x' - \langle x' \rangle\right) \bullet \rho(x, x')dxdx'.$$

- The rms beam emittance is then,

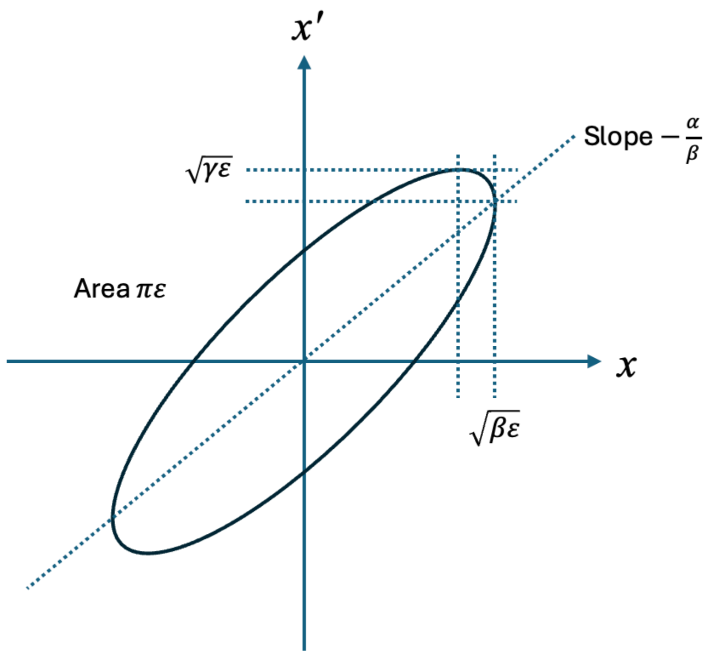
$$\varepsilon_{x,rms} = \sqrt{Det\left[\sigma^2\right]} = \sqrt{\sigma_x^2 \bullet \sigma_{x'}^2 - \sigma_{xx'}^2}.$$

Find $\hat{\beta}$,$\hat{\alpha}$,$\hat{\gamma}$(Twiss parameter) from phase space evolution

- If the beam distribution function is a function of the Courant-Snyder invariant, the σ-matrix is given

$$\begin{pmatrix} \sigma_x^2 & \sigma_{xx'} \\ \sigma_{xx'} & \sigma_{x'}^2 \end{pmatrix} = \varepsilon_{rms} \begin{pmatrix} \hat{\beta} & -\hat{\alpha} \\ -\hat{\alpha} & \hat{\gamma} \end{pmatrix}.$$

$$\varepsilon_{x,rms} = \sqrt{Det\left[\sigma^2\right]} = \sqrt{\sigma_x^2 \bullet \sigma_{x'}^2 - \sigma_{xx'}^2}.$$



$$\hat{\beta}_x = \frac{\sigma_x^2}{\varepsilon_{rms}} \quad \hat{\alpha}_x = -\frac{\sigma_{xx'}^2}{\varepsilon_{rms}} \quad \hat{\gamma}_x = \frac{\sigma_{x'}^2}{\varepsilon_{rms}}$$

Note: $\tilde{K} = \begin{pmatrix} \hat{\beta} & -\hat{\alpha} \\ -\hat{\alpha} & \hat{\gamma} \end{pmatrix}$ is a symplectic matrix, ie $\tilde{K}^T \Omega \tilde{K} = \Omega$

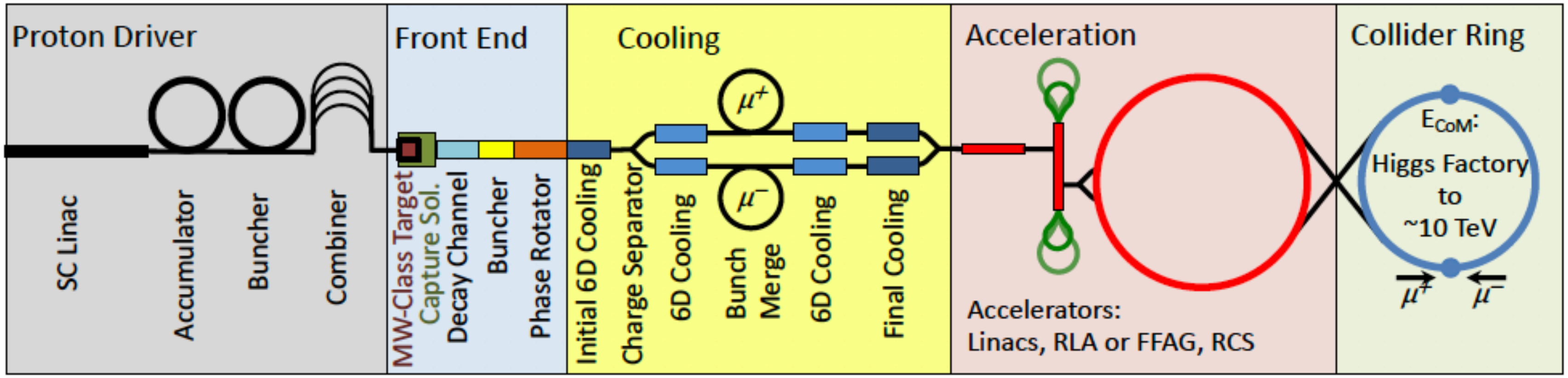
where using $\hat{\beta}\hat{\gamma} = \hat{\alpha}^2 + 1$

Fermilab

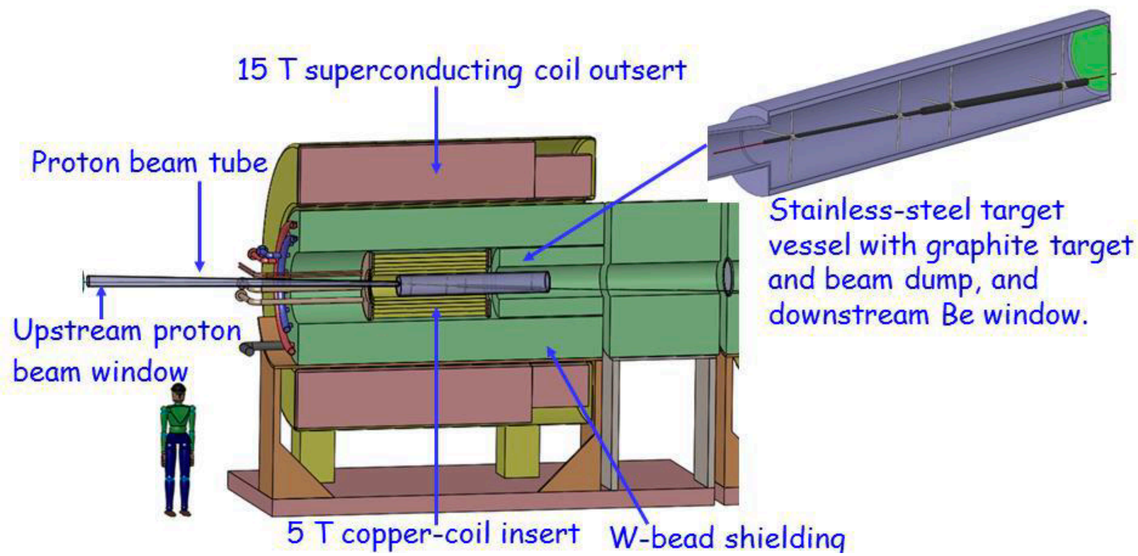
Extra slides

Overview of Muon Collider

Proton beam based muon collider



- $\pi^{\pm}/\mu^{\pm}$ capture magnets



Solenoid base
- Massive & expensive
- Most efficient

Horn base
- Cheap
- Charge dependence

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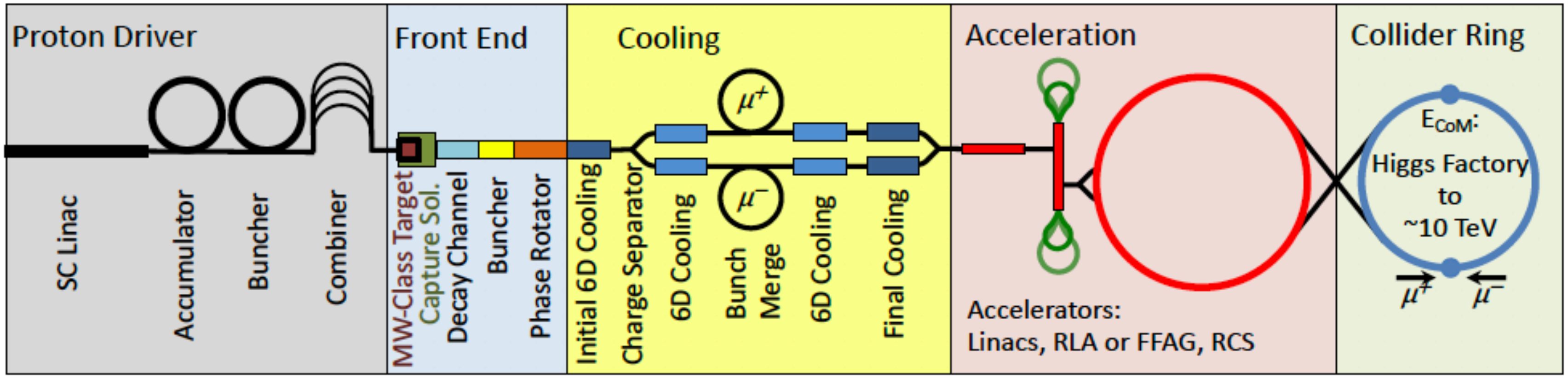
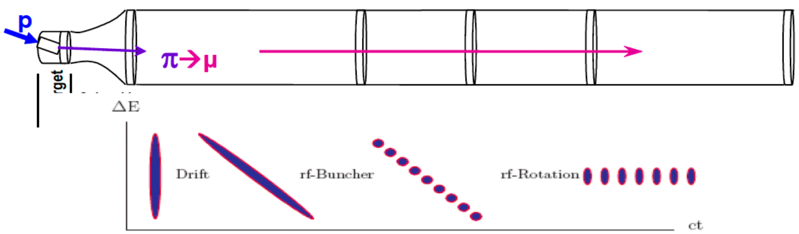


- Buncher and Phase rotator

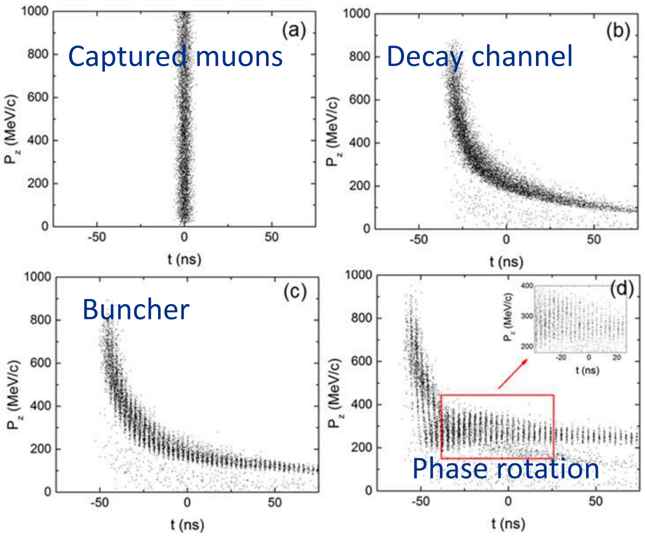
Neuffer



Concept

Simulation

Stratakis et al.

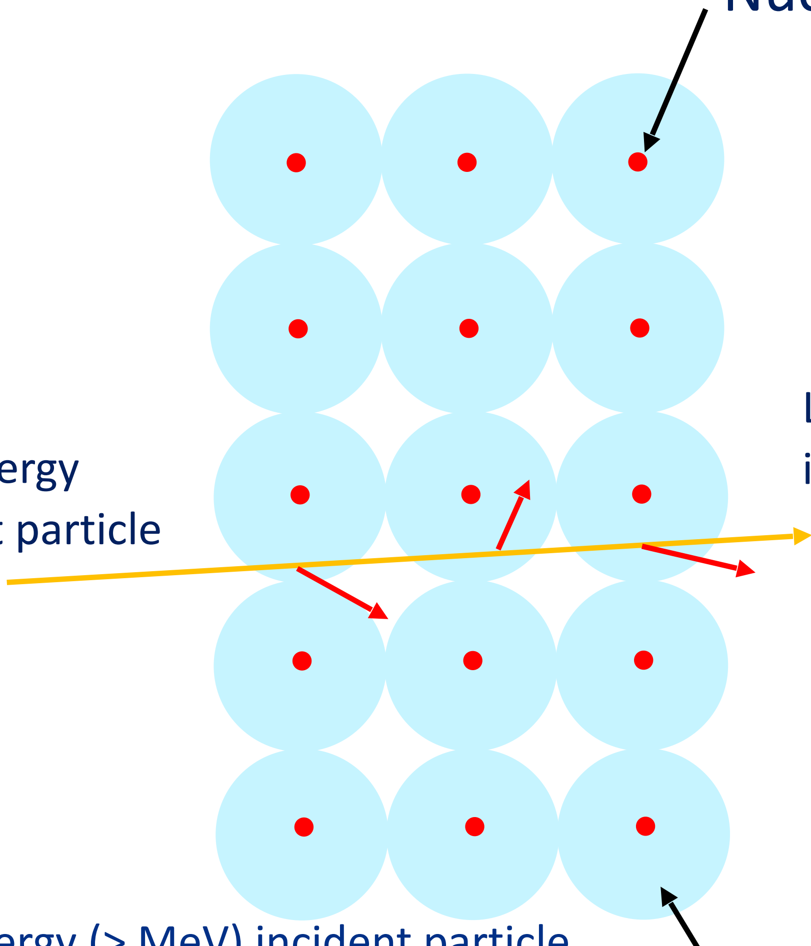


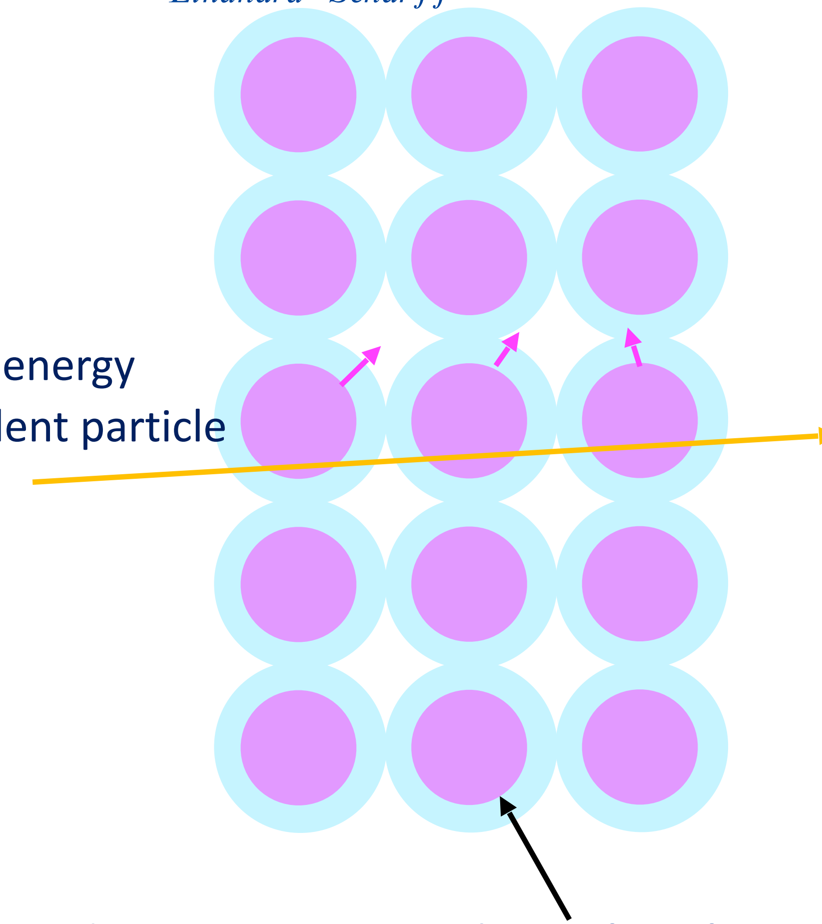
Fermilab

Ionization interactions vs frictional interactions

Low energy (< 100 eV) incident particle feels a screened Coulomb potential

$$S_e(E)_{Lindhard-Scharff} = kE^{1/2}$$

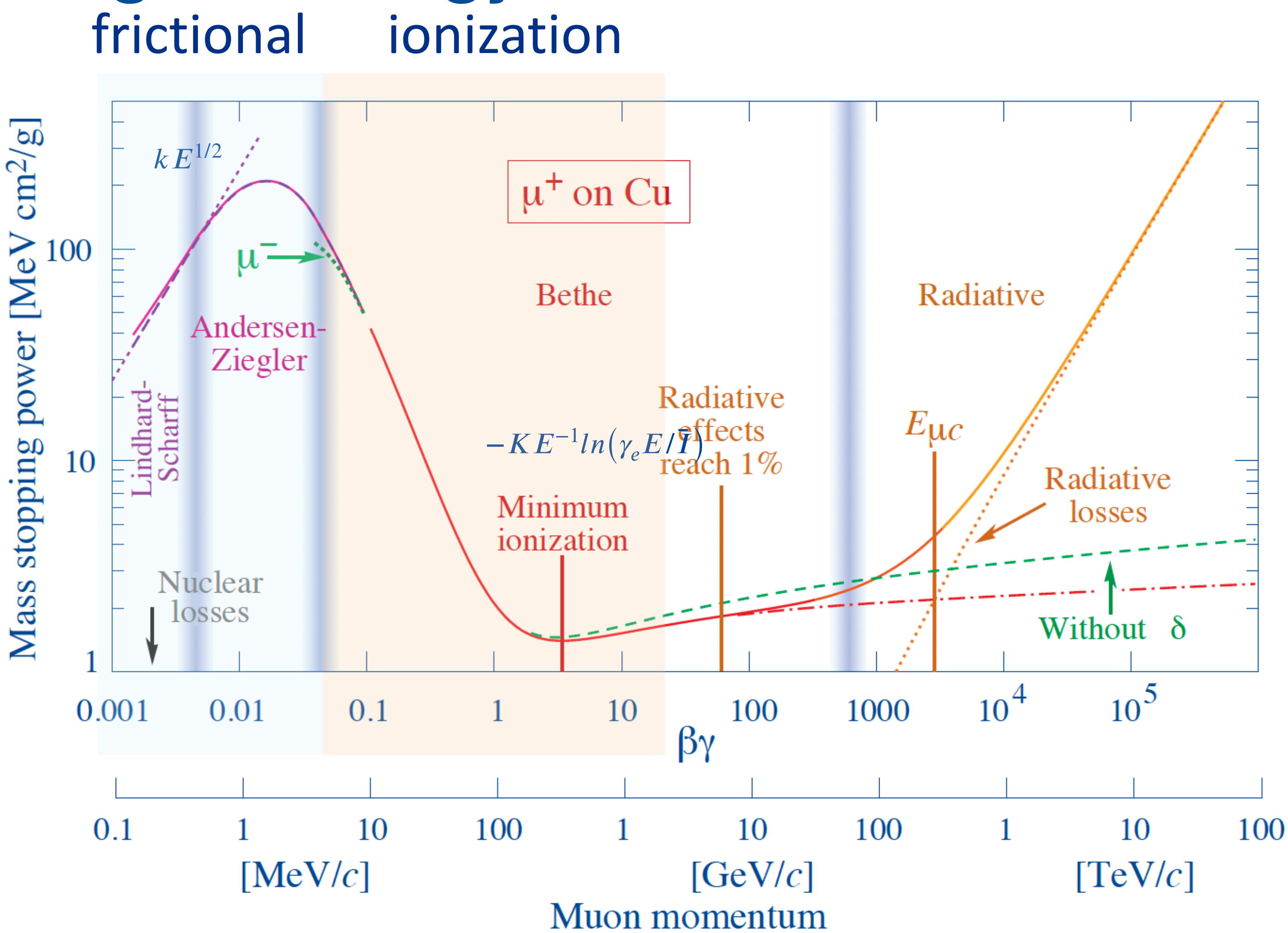




High energy (> MeV) incident particle interact individual atom (electrons)

$$S_e(E)_{Bethe} = KE^{-1} ln\left(\gamma_e E/\bar{I}\right)$$

Wide range of energy loss value in PDG



Muon momentum range is typically 100 ~ 300 MeV/c for ionization cooling (HF solenoid final channel uses p ~ 50 MeV/c)